Humanism, Foundationalism and Modern Mathematics

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ABSTRACT

David Malcolm has suggested that a philosophical account of the nature of mathematical knowledge should cohere with a respect for Biblical revelation. On this basis he argues for a form of intuitionism. I argue that as it is standardly presented, intuitionism clashes with the principle of Man's Stewardship over Nature, but that Malcolm's position escapes this criticism by re-interpreting his intuitionism as a form of predicativism.

Both these positions, and the more common platonist conception of mathematics in opposition to which they were developed, take mathematics to have a foundation — a secure source of basic truths. By arguing that all mathematical knowledge is dependent on language, I argue that the story of the Tower of Babel shows that not only has God divorced our language from reality, but also that the dependence of mathematics on language entails a similar separation in the mathematical case too. Therefore there are no secure 'God-given 'foundations for mathematical knowledge.

There is a standard line in small talk when meeting people for the first time; we ask what they do, where they come from, and so on. This is no less true of Christians than it is of other people. Having recently moved church, I've been meeting a great many believers for the first time, which invariably leads to this string of typical questions. Now, I'm a philosopher, finishing off my doctorate and looking for a teaching job at a university. Invariably, I'm asked if I find that there is tension between my Christian beliefs and my work: a symptom, no doubt, of the anti-intellectualism prevalent in many Western evangelical churches.[1] My usual response is to explain that as I work on the philosophy of mathematics, the stuff I do is so removed from my beliefs, that there will be no clash.

David Malcolm thinks otherwise. Some years ago in this journal, he offered a number of arguments concerning the compatibility of a number of philosophical views about mathematics and a respect for Biblical authority.[2] He contended that even in abstract areas of thought, such as mathematics, our thoughts should cohere with Biblical revelation. Having read his article, I am inclined to agree with Malcolm: even in such an abstract discourse as mathematics, it is important to find an approach which is consistent with Biblical teaching. However, I disagree with him on the details of what would constitute an appropriate approach.

The typical philosophical considerations that arise about mathematics, concern the nature of mathematical objects and our knowledge of them. Three views are traditionally put forward: Platonism, Intuitionism and Formalism.[3] Platonism is the view that ordinary physical objects and numbers are 'on a par'. Numbers are the same kind of thing — objects — as beach balls, only there are more numbers than beach balls and numbers are abstract and eternal.[4] Mathematical statements are true just in case they match up with these abstract states of affairs, and false otherwise. When it comes to normal 'concrete' objects, we always stand in some sort of causal relationship to the things we have knowledge of—yet as they are abstract, mathematical objects are acausal. So this makes our knowledge of mathematics a bit of a mystery, as we are unable to explain the mechanism by which we come to have mathematical knowledge. Formalism and Intuitionism both challenge the correctness of the Platonist account.

According to the Formalist, mathematical statements do not correspond to anything — and instead should be judged in terms of their usefulness, both in terms of unifying and explaining mathematics, and in physical science. On this view, mathematics is like a game we play, and we play the games we do play, rather than some other games, because of the usefulness of playing our games in just the way we do play them. Intuitionism on the other hand, while also arguing that mathematical statements are not made true by some correspondence relation with an abstract mathematical reality, nevertheless argues that mathematical statements are true — because of an innate or in-built ability to count, from which all of our mathematical knowledge stems. Mathematics is thereby a process of creation, and the objects of mathematics are built up of algorithms or proof procedures. On this view, mathematical objects exist if they can be constructed by way of finite descriptions or repeated applications of simple processes.

Malcolm thinks of these attempts to secure mathematical knowledge as a result of humanism about mathematics. I think it is very closely bound up with the notions of foundations, and will call the general approach that these positions share the humanist-foundationalist conception of mathematics. He draws the distinction between a humanist and a Biblically acceptable interpretation of mathematics in the following way:

> To *the humanist, who sees man as the ultimate intelligence, the answer will be along the lines that man is the master of all things, existing and imaginary; therefore man will naturally want to rationalise his world so that he can understand it. Indeed we are quite within our rights to investigate mathematical universes of our own construction.*
>
> *On the other hand, if we believe that the Bible is a divine revelation from the One who created this universe, we will see things differently. We will see man as the prince of creation, as he submits to his Creator's will.*[6]

In the first section, I consider the arguments Malcolm puts forward against these positions, and the positive proposal which he suggests. In the second section, I put forward some independent reasons for rejecting these 'foundationalist' positions. Malcolm's rejection of the various philosophical positions is connected with a rejection of various mathematical practices: I claim that, by rejecting these areas of mathematical study, along with the philosophical positions which offer interpretations of the significance of those areas of study, he is throwing the baby out with the bathwater. The mathematical study of foundations as it is interpreted by the positions he rejects is certainly spurious, but that is not to say that this work should be rejected out of hand. I take this up in the third section, and offer an interpretation of mathematical logic and foundational studies in mathematics, to show that there is a way to study the foundations of mathematics without adding a philosophical interpretation to them. In the fourth section, I put forward some considerations, arguing against Malcolm's theistic foundationalism. As I argued against all forms of foundationalism, in the second section, I go on to offer a philosophical account of mathematics which does not rest on a foundationalist conception. Finally, I mention one or two worries which Malcolm mentions as being motivational to his study, and offer what answers I can give.

MALCOLM'S ARGUMENT

In his paper, Malcolm sets out various detailed criticisms of the three positions outlined above. His main objection to Platonism — which he further divides into two separate approaches, Logicist and Set theoretic — is that it supports a mathematical treatment of the infinite, by accepting the use of 'actual infinite collections'. He concludes that this branch of foundational mathematics — set theory — is irreverent in its treatment of the infinite. Set theory since Georg Cantor (1845-1918) has focused on transfinite sets; that is, infinite collections such as all the natural numbers (1, 2, 3, . . .) or all the real numbers (decimals such as π, 45.6789. . ., 2, etc.). Malcolm argues that it is presumptuous that man can tame the infinite, and so rejects Platonism for attempting a philosophical defence of such practices.

As he — quite rightly — takes there to be strong links between mathematics and its application, Malcolm dismisses the formalist approach: he argues in effect, that if mathematics is just like a game, how is the applicability of mathematics to be explained.

It is worth looking in greater detail at the positive position which Malcolm suggests:

> *'The Intuitionist philosophy comes closest to providing a firm foundation.*[2]

He suggests that the Intuitionist's notions of existence — based on algorithmic construction — might be too extreme, and that perhaps some less extreme criteria be used.

The standard Intuitionist position calls for a revision of mathematical practice; if the mathematicians can only deal with those entities which have already been generated by some method of construction, a statement which refers to entities not yet generated will neither be true nor false, but will have an open true value until such time as the entities in question are generated. This then entails a rejection of the classical principle that for every statement, either it or its negation is true. This 'Law of Excluded Middle' is used in many different areas of classical mathematics, and is one of the easiest ways to characterise those results which are not intuitionistically acceptable.

The revisions that Intuitionism call for, include changing the way in which number theory, real analysis and complex number theory are carried out. Large swathes of classical mathematics just become false, once the principle of Bivalence is rejected. This is something that all Intuitionists accept — they argue that most mathematicians around the globe are misguided, and are literally talking nonsense. Others, such as Everett Bishop, or D. S. Bridges, argue that intuitionistic mathematics is simply another approach to mathematics, with different goals and different methods, and should be tolerated alongside classical mathematics. Either way, there is a problem with Intuitionism: it does not support the sorts of mathematics we require in physics.

Anyone who is familiar with modern quantum theory will know that the phase space of a quantum particle is represented by a 1-dimensional subspace of a Hilbert space; an infinite Hilbert space. Such spaces do not exist, according to the Intuitionists — and there is no way to prove the theoretical results which are used to underpin quantum theory. This might cause some to back down, but the usual intuitionist response is to argue that it will be possible to reformulate the physics in a way that it will be intuitionistically acceptable, and rely only on the more modest resource of intuitionistic mathematics. Recently, an American philosopher, Geoff Hellman, has argued that no such reformulation of quantum theory is possible, as Gleason's theorem — a central theoretical result concerning unbounded linear operators — is not intuitionistically acceptable.[67] The same will hold for any amendment to Intuitionism, which calls for roughly similar revisions of classical mathematics, based on a rejection of the Law of Excluded Middle.

So the Intuitionist is forced to reject Quantum Mechanics — one of the best supported empirical theories today. But such a move is in tension with a well supported Biblical Principle — man's Stewardship of Nature. Malcolm eloquently expresses the role of mathematics and science in this Stewardship:

> *'Mathematics is surely vital to man's God-given function of exercising dominion over this creation. What science could there be without mathematics?'* [8]

There is a line in the philosophy of science, which argues that scientific theories do not have to be true to be useful — physics then would be no more than a game with a usable outcome, but without representational content. This is called Instrumentalism, as scientific theories are reduced to the role of instruments or tools on this view, rather than as truth-apt representations of reality. It would be possible to accept the Intuitionist revision of mathematics, and still accept Quantum Mechanics *in toto,* simply by claiming that while the mathematics was true, the physics need not be. However, I take it that if the Biblical principle of stewardship is strictly adhered to, then our science must be part and parcel of our attempts to understand and care for the world. So our science does have to be more than useful, it has to be taken to be seeking after what is true. So, as the Intuitionist revision of mathematics entails revising physics, and as the Christian has no Instrumentalist 'opt out clause', revising physics is the only option available once Intuitionism is accepted. But this is a move that the Christian should not make, so standard Intuitionism should not be embraced.

The position Malcolm is seeking then, would have to be something strong enough to support a larger proportion of classical mathematics, but which still begins with an innate intuition of the natural numbers, and which in some sense, constructs the rest of our mathematical knowledge from such a foundation. As we shall see in Section 3, Malcolm is also suspicious of the modern treatment which transfinite sets receive, and therefore might also want to make some restriction in this area too.

One position meeting all such requirements is Predicativism, seen by some as the big brother of Intuitionism. The position was first put forward by Henri Poincare, who was one of the most influential and important mathematicians of the century. His philosophical position is subtle and complex, and is only now receiving the attention it deserves. Like the Intuitionist, the Predicativist restricts the operations which may be performed on mathematical objects, and so does not embrace all of classical mathematics: but the Predicativist's restrictions revolve around the notions of impredicativity and quantification. An impredicative definition refers to a collection which contains the object to be defined. The Russell set, for example, is the set of all sets which do not contain themselves. The problem with such principles is this: if the definition is thought of as constructing the defined objects, then impredicative definitions are circular.

Rejecting such impredicative methods, the Predicativist is able to hang onto classical principles such as the Law of Excluded Middle, and any of the results which rely on this law, but is able to reject, for example, the multiplicity of transfinite or 'infinite' sets which the classical mathematician accepts.

FOUNDATIONS AND FOUNDATIONALISM

For most of this century, philosophers have concentrated their attention on a particular conception of knowledge — foundationalism. Descartes is often thought of as the founder of this approach to knowledge; he tried to strip away all of the beliefs he held which were open to doubt, and thereby arrive at an incorrigible and indubitable foundation, from which he would be able to rebuild all of his genuine knowledge claims. His principle foundational units were the *Cogito* — by which he knew he himself existed — and his belief that God exists, and as God is good, would not let him be overwhelmingly deceived. Later philosophers have argued that this foundation should also include our sense perceptions, memories and introspections, and have argued for various ways in which these units build up to justify all of our genuine knowledge claims. This view led to the belief that unless it were possible to establish some secure foundation for a given subject, then all knowledge claims in that area were open to doubt.

Foundationalism rests on the belief that we can step out of all of our beliefs and ways of assessing the credibility of our beliefs, and examine them all afresh. It requires some sort of external standpoint — in short, it requires us to have God's view of our intellectual development. One of the clearest analogies is that of knowledge as a ship, and knowledge as a house. If our knowledge is like a house, then indeed we can step outside and examine it, and we can look into the state of the foundations. But knowledge is not like a house — for we can never leave it. Instead, it is more like a ship, which we can examine while we sail, but can never put in to dry dock. We can make running repairs as we go, but we can never take that step back to examine her from a separate perspective.

Despite the growing credence of this non-foundational approach to epistemology (the study of knowledge), foundationalism has fought a rear-guard action in the philosophy of mathematics. The main reason for this is that, unlike many other subjects, mathematics does have a clear and well defined notion of foundational studies. Through the course of the last century, mathematicians were slowly realising that the intuitive notions they were using in their proofs were not sufficiently rigorous. Various counter-intuitive results had been proven, and so there began a general trend to make the subject more precise and more rigorous. This was most noticeable in real analysis and the study of the Calculus: Newton and Leibniz had both thought in terms of infinitesimals, and based their notions of derivatives and integrals on changes over infinitesimals. Rather than rely on the existence of such infinitely small numbers, Weierstraß produced the now accepted ε-δ notation, based on the notion of an arbitrarily small quantity. This general trend is often called the 'arithmetisation of analysis', as many concepts in real analysis (Calculus) were reduced to concepts that only required a grasp of arithmetic. This led various mathematicians to try to put arithmetic on a more rigorous footing — to give a definite and precise treatment of our informal and everyday use of arithmetic. Peano, Dedekind and Frege all produced apparently acceptable formalisations of arithmetic — the axioms looked to be true of arithmetic and on the whole they were thought to measure up to informal practice pretty well. However, Bertrand Russell discovered that Frege's system[9] was inconsistent — it contains a contradiction. What looked like the most secure and certain area of knowledge, was suddenly up for doubt, and the search began to find an alternative foundation for mathematics.

Even though historically, foundationalism was not the drive to discover axiomatisations for arithmetic, it nevertheless soon took over and became a dominant factor in promoting the study of foundations. Philosophical positions such as Platonism, Intuitionism and Formalism arose as support for various types of foundation that were put forward — Platonists on the whole advocating set theory, Intuitionists recursion theory, and Formalists the study of formal systems. And certainly, man wanting to be master of his own house — humanism — is an additional factor at play here. I think it is this combination of humanism and foundationalism which are Malcolm's real targets — although he does not express his point this way.

I would want to argue against the foundationalist conception of the philosophy of mathematics, simply on general grounds: foundationalism has been largely discredited as a philosophical doctrine. I therefore feel a burden of proof to explain the significance of foundational studies, such as set theory and mathematical logic quite generally, to explain the role of 'foundations without foundationalism'. I'll say more about this in the next section.

First however, I should say something about Malcolm's attitude to foundationalism. Reading his paper, it becomes clear that he thinks mathematics both needs and has a foundation — in an innate grasp of the mathematical facts, which is God given. As such, his rejection of humanism-foundationalism is different from mine: he thinks that these philosophies are barking up the wrong tree; I'm denying that there are any appropriate trees in this area to bark up.

MATHEMATICAL LOGIC, SET THEORY AND THE FOUNDATIONS OF MATHEMATICS

In his paper, Malcolm brings forward criticisms of two foundational projects — set theory and attempts to formalise arithmetic. His criticisms are briefly this: that set theory concerns infinite sets, and

> *'to actually formalise a treatment of the infinite, would therefore seem to be a case of man exalting himself to a position of omniscience.'*[8]

As he takes arithmetic to be self-evident, and God-given, he objects to the attempts by mathematicians, to give a formal theory of arithmetic, and prove it consistent.

As I have claimed above, the original attempts to formalise arithmetic were not related to trying to show that arithmetic was certain, but were the results of a general raising of professional standards in the subject. Set theory likewise had origins quite separate from the foundational disputes with which it is now linked. There is a common received history of these problems, which is recited in books on the history of mathematics, which distort the actual events, and cause no end of problems when philosophers of mathematics discuss these problems with mathematicians: so, let me review some of the history of set theory.

According to Euclid, a limited plurality *(plethos horizomen)* is such that the sum is greater than the parts.[10,11]

An unlimited collection then, is one where some proper part is equal in size to the whole collection. Dedekind used this in 1872 as a definition of an infinite collection: take for example the natural numbers *N*. Then the even numbers, denoted *2N,* are a proper subcollection of the natural numbers, yet there are just as many even numbers as natural numbers.[12] Let us say that a set is Dedekind-finite if every proper subset is smaller than the set itself: it has a smaller cardinality.

In the same year that Dedekind was proposing his definition of finitude, a close friend of his — Georg Cantor — discovered that there were many more real numbers than there were natural numbers: any attempt to equate the two collections left as many reals unaccounted for as there were reals in the first place. Cantor showed that the rational numbers — the proper fractions such as $2^3/_4$, $5^1/_2$, $^{42}/_{67}$ etc. —were equinumerous with the natural numbers: he decided that sets of this size have the cardinality $\aleph_0$ (aleph-null), his first transfinite number. He wondered whether the cardinality of the reals, *c*, was the next transfinite cardinal number $\aleph_1$. Cantor postulated an uncountable number of these alephs, and devised a system of cardinal arithmetic for them, similar to, but not the same as, natural number arithmetic. Cantor is often thought of as introducing the notion of an 'actual infinite set' into mathematics — reading Cantor, he said no such thing. He never claimed to be dealing with infinite collections. Certainly, the collections he deals with are Dedekind-infinite, and so they are often referred to as 'infinite collections', but why should we take Dedekind's word on the matter? Cantor argued that if a set has a definite size, then it is finite. As the sets he describes have a definite size, they must be finite by this definition; as they go beyond the traditional conception of the finite, he called them transfinite sets.

Malcolm objected to Cantor's treatment of such sets on the following basis:

> '[S]*urely mathematics should only embody concepts we can understand, and God has clearly not allowed us to comprehend infinity.*'[8.]

But this is exactly the argument that Cantor used to defend his work: we do understand transfinite mathematics: ergo, transfinite mathematics does not deal with the genuine infinite. Set theory is a rich source of new results, which filter through to mathematicians working in more usual areas — and often these mathematicians talk of 'infinite sets' simply because the notion of Dedekind-finiteness has become so entrenched. But I think this is no more than a linguistic mistake — I do not think that it should be used as a stick with which to beat the set theorist.

How did set theory become involved in the foundationalist disputes anyway? After the collapse of Frege's formal system of arithmetic, mathematicians tried all sorts of ways to provide a foundation for arithmetic. Set theory was one way that was tried; it is possible to represent natural numbers as sets, taking the empty set $\varnothing$ to represent zero, and then the number of things that are zero, that is, $\{\varnothing\}$, to be one, and then $\{\varnothing, \{\varnothing\}\}$ to be two, etc. It can be shown that using these definitions, Peano's axioms can be modelled consistently in set theory, and so it is often claimed that this provides a foundation for arithmetic.

Malcolm argues:

> '[T]*he point is: is a proof needed for the elements of arithmetic? I suggest that a sufficient proof lies in the fact that all mankind understands it.*'[8]

If the foundationalist is to be believed, then a proof is needed. As Malcolm seems to favour some form of foundationalism, albeit an anti-humanist form of foundationalism, then it is natural that he should take a theistic justification of arithmetical foundations.

To ask if a proof is needed for arithmetic, is to ask the wrong question. The foundationalist claims that if arithmetic is to be a source of certain knowledge, then a proof must be given that its foundations are secure; for example, that the axioms are consistent — they contain no contradictions; and complete — they generate all of the theorems of arithmetic. So the proof is relative to a purpose. Malcolm, in challenging that purpose, is willing to accept a different standard of evidence; no mathematical proof would ever convince him, as he seems to hold that a non-mathematical justification of the certainty of mathematical knowledge is required.

On the other hand, I'm more concerned with questions of epistemological adequacy. Arithmetic has been around for a long time — but it is only in the past century or so that attempts have been made to formalise it rigorously. I'm much more concerned with whether the axioms we give for arithmetic are faithful to existing practice. As Frege's system set out with such aims, it is not at all obvious that when we try to be precise, we will not make mistakes. Just because mathematics deals with necessary truths, does not mean that all of our mathematical statements are necessarily true. So when presented with an axiom system that claims to represent some already well established practice, I want to know whether these axioms respect the conceptual foundations already in place in the informal practice; are the axioms consistent, and do they adequately express all of the results in the informal practice. As they are formalised, these axioms now can give a clear insight into how various concepts are related, and which concepts play a role in which results — an entirely different role for foundational studies than is presented by the foundationalist.

I would argue that while Malcolm is right to reject the humanist-foundationalist conception of foundations, he should not reject foundational studies outright. His rejection is based on a rejection of humanism; mine on a rejection of foundationalism. I would argue that even with his brand of 'theistic' or innate-foundationalism, the notion of mathematical foundations I advocate is harmless. I hope I have shown ways that sidestep his objections to these areas, and that his objections are too closely tied to the foundationalist conception of foundations.

FOUNDATIONS —GOD-GIVEN AND INNATE

Typically, Christians have thoughts about mathematics in the following way:

> *'In pure mathematics we contemplate absolute truths which existed in the Divine Mind before the morning stars sang together, and which will continue to exist there, when the last of their radiant host shall have fallen from heaven.'*[13]

The assumption is that God created mathematics, in the same way that He created the rest of the world, and that mathematical knowledge is a process of discovery. Such a thought strikes me as at best, overly optimistic. We live in a fallen world, a post-Babel world, and even if there is a language of Heaven, and a single, unique description of the world — physical and abstract — we no longer have access to it.

We speak with earthly tongues, which we must learn. As a child, my parents would point out objects in the world and name them, or the properties they possessed. At a certain stage, based on these finite observations, I started to name things in ways that conformed to these experiences. To those around me, if I used the words in the way that they would, then I had grasped the meaning correctly, and shared the same understanding as they had; else if I used a word in a way different to them, they would say that I did not properly understand the meaning.

But think about what is really going on: given a finite number of examples, I was trying to carry on a sequence, trying to find a rule to follow. But, there are any number of ways to carry on the initial segment of a sequence, any number of rules to generate more members of a sequence. My manifestation of meaning does not show that I share the same meaning-rules as those around me, only that my use coincides with their use.

The moral is this: we have no contact with mathematical facts except through our language. Mathematics is **linguistically mediated.** As we live in a post-Babel world, the best we can do is communicate: we cannot be certain that we possess the same meanings — what Schaeffer would attribute to our lack of universals.[14] As a result of this, there is no privileged access to mathematics; we cannot assume that just because it is an abstract subject matter, that our contact with the realm of mathematics is the same contact as God would have; I repeat, there is no privileged access to mathematics.

Taking Babel seriously, the thought is that once we may have had genuine direct access to the appropriate language with which to describe reality, including mathematical reality. But God has cut the ties between our language and reality — that is the message of the Tower of Babel. We no longer have a guarantee that when we form concepts, we will latch onto the important constituents of reality. This goes for all of our concepts — those about the physical world as well as those about mathematics. So we cannot help but approach mathematics from a distinctly human perspective. At its best moments, our language tracks features of the world; at other times, it is a matter of projection onto the world of various aspects which enable us to interpret and understand what is happening around us. The language of Heaven may give a unique and accurate description of the world — but we have no access to that language.

ANTI-FOUNDATIONALISM

In the second section, I tried to argue that there are legitimate ways of studying the foundations of mathematics, without seeing such an enterprise as a search for epistemological security. In this section, I'd like to develop an account of mathematics, which takes into consideration the thought that we can get our mathematics wrong, and that we can be mistaken in our mathematical beliefs. So it is a form of fallibilism.

Not that I think that we are often wrong about the mathematics we do, but I think that we should admit that this is always a possibility. Foundationalism tried to build up, in one way or another, from various simple units, true beliefs which we are justified in holding, because they are related in the appropriate fashion to the foundational units which we are justified in accepting. Rather than start with the thought that our basic beliefs are justified, one approach to a non-foundational theory of knowledge is to argue that basic beliefs do not fall into the category of justified or unjustified beliefs. To justify any one of our beliefs, we must measure it against the standards of acceptability which we use to judge whether it is justified or not.

The conceptually basic units of any discourse are so closely linked with those standards of rational acceptability, that judging them would be like measuring a yard-stick. It makes no sense to measure our measuring apparatus.

Obvious examples of thought upon which the rest of our knowledge hinges, include our belief in the external world, the reliability of our senses, and the uniformity of nature. We can certainly question these hinge thoughts once we have a developed body of knowledge and beliefs, but in such questioning, we are tacitly moving to new standards of measurement — and so to new hinge thoughts — from which to appraise these ideas.

In mathematics, various hinge thoughts seem to operate — for example, the reliability of arithmetical induction and of logical proof. When a subject comes under mathematical investigation, the methods of modern mathematics are to formalise the concepts implicit in the informal practice; these informal concepts form a hinge — an axiomatisation is accepted just in case it measures up to the standards expressed by the hinge propositions, in which case the axiomatisation is thought of as a faithful representation of the heuristic content.

There are two ways in which an axiomatisation can fail: logically and heuristically. A logical falsifier points out some inner contradiction — such as Russell pointed out to Frege, when he explained the contradiction which arises from consideration of the Russell set (mentioned above). Heuristic falsifiers can be just as damaging, but tend to be less dramatic — they are counter-examples which show that the axioms fail to fully capture some point of the heuristic content of the informal practice.

In his 1988 paper, Malcolm suggested that something along the lines of Intuitionism might be tenable: I have argued that due to the fall of man, the language with which we express our mathematics is fallible. The sort of position I would put forward myself takes this into account. I espouse something a bit like Formalism — call me a neo-formalist.

I think that whenever we put forward an axiom schema, and investigate its properties, we are doing something worth calling mathematics. Arguably, this can involve little more than symbolic manipulation, and as such, is merely a rule-governed activity or game. I think that such a practice embodies standards of correctness which are worth describing in terms of truth and falsity — although this is a very minimal notion of truth.

When the axioms schema also contains heuristic content — where it is supposed to represent something, and is supported by a hinge proposition — then I take it that the mathematics is up for being true or false in a more robust sense, that is, in the sense of corresponding to something. The usefulness of mathematics is explained by the appropriateness of the various heuristic contents expressed by the axioms of mathematics. These axioms are not up for proof, but are accepted because they express concepts which we already value.

FORMAL AND INFORMAL MATHEMATICS

In the conclusion to his article, Malcolm raises a worry which, I confess, I cannot see the weight of. He writes:

> *'No satisfactory proof has ever been devised to show that the basic number theory is universally true; rather proofs have been put forward indicating that such a proof may be impossible. Thus mathematics cannot contradict Scriptural teaching that God exists as three persons in unity.'*[15]

The worry seems to be roughly this: that if we take the Bible literally, then it would seem to be in contradiction with our mathematical practice. Rather than follow any of the numerous theological solutions which have been advanced to explain this problem, Malcolm turns to mathematics — to wonder whether perhaps mathematics might not be universally true, but merely true for mankind.

This is where his foundationalist epistemology shows through — he argues that if one of the other positions (Platonism, Formalism or Intuitionism) had proved the axioms of mathematics, then the doctrine of the Trinity would be in serious doubt. As he thinks that God has put the innate knowledge of mathematics inside us, there can be no contradiction in this area.

What I do not understand, is why Malcolm takes this to involve peculiarly modern mathematics. Scholars have been trying to explain the Trinity for centuries — as it stands, at least on a superficial level, it **does** contradict our mathematical practices. The formal treatment which mathematical concepts receive in modern mathematics, aims to be faithful to the informal notions at play in our past mathematical practice — so I cannot see why Malcolm takes this to be a problem posed by the combination of Biblical authority and modern mathematics; and why I also cannot see why any solution could revolve around casting doubt on the axioms of modern mathematics, as this would not solve the problems raised by pre-axiomatic mathematical practice.

CONCLUSIONS

To sum up: Malcolm has argued for a rejection of the traditional positions in the philosophy of mathematics; he argues that these positions all share a common humanist-foundationalist attitude, which he rejects. He argues that this attitude also infects the mathematical study of the foundations of mathematics, in particular, the study of *'actual infinite collections'*: so foundational studies should also be rejected. He argues instead that we should think of mathematics as founded on a God-given innate grasp of mathematical truths.

I have tried to argue that while I also reject the humanist-foundationalist approach, I do so on different grounds: I reject foundationalism. But despite my rejection of this epistemological doctrine, I see no reason to reject the study of the foundations of mathematics, as I can see independent reasons for such study, and have tried to argue that such independent motivations should also make this area of mathematics acceptable to someone taking a position along Malcolm's lines. In particular, I think that the term *'actual infinite collection'* is a misnomer — Cantor, who introduced this way of thinking into mathematics, talked about these sets as being 'transfinite' and saw them as an extension of our current notion of 'finite', rather than as an attempt to grasp the ungraspable.

As I do not accept any form of foundationalism, I try to show that even Malcolm's admirable combination of what is essentially a form of Predictivism, coupled with his form of 'theistic' foundationalism, is philosophically misplaced. Mathematics is an essentially linguistic practice, and post-Babel, we have no reason to think that our language latches onto reality in the way that we intuitively think it does. So even if it appears that we have an innate grasp of mathematical truths, as we live in a fallen world, we should not assume that these innate beliefs are incorrigible.

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