Critical analysis of Humphreys' shell metric cosmology

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It is shown herein that the Humphreys' solution for the metric inside a thin shell of matter is incorrect. The interior metric is independent of time and there is no 'timeless' zone in the interior. The interior of the shell is flat and isometric to Minkowski spacetime. We expect this as the interior of the shell has no gravitational field, as in the Newtonian limit. It is also shown herein that Humphreys' assertion of new solutions of the Einstein field equations (EFE) is incorrect. The metrics he presents do not solve the EFE. In fact, one form of the solution is obtained by a major mathematical error. Thus, the conclusion herein is that the shell model is fatally flawed and does not provide a solution to the starlight travel time problem.

In papers[1,2] published in the *Journal of Creation*, Russ Humphreys makes several extraordinary claims about a new solution to the EFE of conventional general relativity (GR). One such claim is the existence of 'timeless' zones.[2] It is the purpose of this paper to show that, contra Humphreys, spacetime in the interior of the shell is flat and *static* and is isomorphic to a subset of Minkowski space. As a result, clocks inside the shell are inertial and measure proper time. Pulsations in the radius of the shell as a function of time have no local effect on clocks in the interior. We expect this from the equivalence principle and the Newtonian limit of GR. In GR the idea of a constant gravitational 'potential' function inside of the shell is not relevant and can lead to mistaken conclusions. This is the case in the manner in which Humphreys imposed continuity on the metric and then induced a time-dependent 'potential' inside of *R(t)*. Humphreys provides no justification of his continuity requirement, other than a suggestive appeal to a putative Newtonian gravitational potential. Regarding these two claims, we show that continuity of metric components is not a requirement for connected manifolds. Moreover, extreme time dilation is a relativistic effect and is the regime in which Newtonian physics is completely inapplicable. As this note will show, Humphreys' requirement is at odds with the formalism of GR and also ignores the fact that the Newtonian potential, which is illegitimately imported into GR, is only defined up to an arbitrary constant. Humphreys' conclusions of 'stopped clocks' and 'timeless zones' in the interior are mistaken.[2] His new arguments repeat the same mistaken interpretation of the time coordinate as is evident in his original ideas, published in *Starlight and Time*.[3] In the course of this paper, two solutions for an inhomogeneous solution with an interior cavity are presented. Both solutions show there is no 'timeless zone' in the cavity.

Also, it is shown below that Humphreys' assertion of new solutions of the EFE is incorrect. The metrics he asserts do not solve the EFE. In fact, one form of the solution is obtained by what I will demonstrate to be a major mathematical error.

This error appears to be instrumental in reinforcing Humphreys' notion that his new 'principles' of metric continuity and Newtonian potentials in GR are correct.

Finally, it should be noted, the shell configuration that Humphreys proposes was correctly solved by W. Israel.[6] That solution appears as exercises 21.25–27 in MTW.[7] The solution to that exercise shows that Humphreys' position is incorrect. While the solution could be presented, it is felt that the following critique would be more accessible to readers without extensive knowledge of differential geometry techniques. Those who have such knowledge are referred to Israel[6] and MTW.

Notation conventions are $c = 1$, and $d\Omega^2 = \left(d\theta^2 + \sin^2\theta d\phi^2\right)$

Derivation of the interior and exterior metric of a thin shell within GR

Contra Humphreys,[1] the interior of the spherical shell is static and flat (isometric to Minkowski space). I follow Synge.[8] In this section I derive the correct metric of a shell by staying within the theory of GR. I point out later that Humphreys' condition overlooks the fact that Newtonian potentials are only defined up to an arbitrary reference value.

With a slight change in notation we write the metric as (cf. Synge (1971), p. 270):

$$ds^2 = -e^{\gamma}dt^2 + e^{\alpha}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right) \qquad (1)$$

The functions α and γ are functions of the coordinates $r = x^1$ and $t = x^4$. The pertinent Einstein equations reduce to (cf. Eq. 81, p. 282 in Synge):

$$
\begin{aligned}
G_1^1 &= r^{-2} - r^{-2}e^{-\alpha}\left(1 + r\gamma_1\right) \\
G_2^2 &= G_3^3 = e^{-\alpha}\left(-\frac{1}{2}\gamma_{11} - \frac{1}{4}\gamma_1^{\,2} - \frac{1}{2}r^{-1}\gamma_1 + \frac{1}{2}r^{-1}\alpha_1 + \frac{1}{4}\alpha_1\gamma_1\right) \\
&\quad + e^{-\gamma}\left(\frac{1}{2}\alpha_{44} + \frac{1}{4}\alpha_4^{\,2} - \frac{1}{4}\alpha_4\gamma_4\right) \\
G_4^4 &= r^{-2} - r^{-2}e^{-\alpha}\left(1 - r\alpha_1\right) \\
e^{\alpha}G_4^1 &= -e^{\gamma}G_1^4 = -r^{-1}\alpha_4
\end{aligned}
\tag{2}
$$

G_b^a are the components of the Einstein tensor, subscripts on α, γ denote partial derivatives with respect to the coordinates $r = x^1$ and $t = x^4$.

The third equation in (2) can be integrated in terms of G which is directly related to the stress-energy tensor. Cf. equation (83) p. 273 in Synge:

$$e^{-\alpha} = 1 + \frac{8\pi}{r}\int_0^r r^2 T_4^4 dr + \frac{C(t)}{r} \tag{3}$$

In eq.(3) we have substituted $G_4^4 = -8\pi T_4^4$ and included the term which depends on the time dependent 'constant' of integration, $C(t)$. Note that $C(t)=0$, since otherwise the metric is singular at the origin, which would be unphysical as there is no matter at the origin.

We now integrate the equation along a *t=constant* surface, using a stress-energy for a shell instantaneously at $r=R(t)$, for which:

$$T_4^4 = -\frac{M\delta(r-R)}{4\pi r^2} \tag{4}$$

(Note $\delta(x)$ is the Dirac delta function.) This yields the result:

$$e^{-\alpha} = \begin{cases} 1 & \text{if } r < R(t) \\ 1 - \dfrac{2M}{r} & \text{if } r \geq R(t) \end{cases} \tag{5}$$

We now use this to integrate the first (G_1^1) of the equations (cf. equation (84) in Synge (1971))

$$
\begin{aligned}
\gamma &= \int_0^r \left(\frac{e^{\alpha}-1}{r} - re^{\alpha}G_1^1\right)dr \\
&= \int_0^r \left(\frac{e^{\alpha}-1}{r} + 8\pi re^{\alpha}T_1^1\right)dr
\end{aligned}
\tag{6}
$$

For the interior where $r < R(t)$, with $T_1^1 = 0$, yields

$$\gamma = \gamma(0) = 0. \tag{7}$$

The choice of $\gamma(0) = 0$ is due to the stipulation (boundary condition) that the t coordinate is the time measured by a clock at rest at the origin $r=0$. Such a clock is inertial and will register the proper time $dt = \sqrt{-ds^2}$. Comparing this with equation (1) requires $\gamma = 0$ or $g_{44} = -1$ for $r < R(\tau)$.

Rather than setting up a stress energy tensor for the shell at this point, in order to integrate beyond $r = R(t)$, we note that by Birkhoff's theorem the solution exterior to the mass shell is the Schwarzschild solution. Therefore,

$$
\begin{aligned}
ds^2 &= -dT^2 + dr^2 + r^2 d\Omega^2 && \text{if } r < R(\tau) \\
ds^2 &= -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 && \text{if } r > R(\tau)
\end{aligned}
\tag{8}
$$

Equations (5) and (8) show that the interior solution is flat, static Minkowski space (empty space with no gravitational effects) and that the metric local coordinate components are *discontinuous*. This is the same solution as given by W. Israel.[6] Note that T is the physical (proper) time of inertial clocks in the cavity; and t is the physical (proper) time of inertial clocks at 'infinity'. In particular $T \neq t$ and equation (8) shows the two charts that cover different regions of the solution. Appendix A below provides the embedding equations that demonstrate the consistency of the solution. This solution is what we would expect from first principles of GR and the Newtonian limit as the spherical shell can have no physical effect on the interior (there is no gravitational field in the interior). From equation (5), above, the metric is not continuous but has a step discontinuity at the shell. This is expected from first principles. This shows that Humphreys' *ad hoc* requirement of a 'continuous Newtonian potential' boundary condition for the metric is mistaken. *It is a misapplication of Newtonian concepts in a general relativistic domain where they do not apply.*[9] We revisit the continuity requirement in section 10, below; again, showing that the requirement is false.

In closing, we see that this result could be obtained directly from Birkhoff's theorem. The interior spherically symmetric solution must be static, and since there is no central mass, simply setting $M = 0$ in the Schwarzschild solution yields Minkowski space in the interior. Also, we can simplify issues of the stress-energy tensor by solving the EFE in comoving coordinates. We do this below, from which we will again see that the interior is flat Minkowski space. The main point of this section was to derive the discontinuity of the metric components and to establish that the interior is Minkowski from a simple integration of the EFE.[10]

Claim of an exact solution is false

Humphreys[1,2] claims that his metric is an exact solution of the EFE. That claim was not shown rigorously and Humphreys just seems to have thought it was a solution since Φ was assumed to be constant. It is trivially so if Φ is a constant independent of both position and time. Then in an unproven generalization, he makes *R* a function of time, which violates the assumed constancy. Φ then becomes a function of time. I write his metric here:

$$ds^2 = -\left(1+2\Phi(t)\right)dt^2 + \left(1+2\Phi(t)\right)^{-1}\left(dx^2+dy^2+dz^2\right) \quad (9)$$

Using Synge's space-like sign convention rather than Humphreys' time-like sign convention for ease of analysis (the sign convention does not alter the physics), rewriting in polar coordinates gives:

$$ds^2 = -\left(1+2\Phi(t)\right)dt^2 + \left(1+2\Phi(t)\right)^{-1}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right)\right] \quad (10)$$

In this form we can immediately employ Synge's equations for the EFE in *isotropic coordinates* and compute the mixed Einstein tensor. The Synge metric, equation (70) p. 270, is:

$$ds^2 = -e^{\gamma}dt^2 + e^{\alpha}dr^2 + e^{\beta}\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad (11)$$

Comparing (10) with (11), respectively, we get the identifications:

$$\begin{aligned} e^{\alpha} &= \left(1+2\Phi\right)^{-1} \\ e^{\beta} &= r^2 e^{\alpha} = r^2\left(1+2\Phi\right)^{-1} \\ e^{\gamma} &= \left(1+2\Phi\right) \end{aligned} \quad (12)$$

$$\begin{aligned} \alpha &= -\log\left(1+2\Phi\right) \\ \beta &= \alpha + 2\log r = 2\log r - \log\left(1+2\Phi\right) \\ \gamma &= \log\left(1+2\Phi\right) \end{aligned} \quad (13)$$

Next, we evaluate the partial derivatives needed for the mixed Einstein tensor (Synge (1971), equation (78), p. 272). This gives the following non-zero factors:

$$\begin{aligned} \alpha_1 &= 0 \\ \alpha_4 &= -2\dot{\Phi}\left(1+2\Phi\right)^{-1} \\ \beta_1 &= \frac{2}{r} \\ \beta_4 &= \alpha_4 = -2\dot{\Phi}\left(1+2\Phi\right)^{-1} \\ \gamma_4 &= 2\dot{\Phi}\left(1+2\Phi\right)^{-1} \end{aligned} \quad (14)$$

The needed second derivatives are:

$$\begin{aligned} \alpha_{44} &= -2\ddot{\Phi}\left(1+2\Phi\right)^{-1} + 4\dot{\Phi}^2\left(1+2\Phi\right)^{-2} \\ \beta_{11} &= -\frac{2}{r^2} \\ \beta_{44} &= \alpha_{44} \end{aligned} \quad (15)$$

To show that the metric is not a solution, we need only show that one component of the Einstein tensor is not zero. The energy component G_4^4 is easiest and will suffice. Using the third equation of the equations (78) in Synge (1971) gives:

$$\begin{aligned} G_4^4 &= e^{-\alpha}\left(-\beta_{11} - \frac{3}{4}\beta_1^2 + \frac{1}{2}\alpha_1\beta_1\right) + e^{-\beta} + e^{-\gamma}\left(\frac{1}{4}\beta_4^2 + \frac{1}{2}\alpha_4\beta_4\right) \\ &= (1+2\Phi)\left(\frac{2}{r^2} - \frac{3}{4}\left(\frac{2}{r}\right)^2\right) + \frac{1}{r^2}(1+2\Phi) \\ &\quad + (1+2\Phi)^{-1}\left[\frac{1}{4}\left(\frac{-2\dot{\Phi}}{(1+2\Phi)}\right)^2 + \frac{1}{2}\left(\frac{-2\dot{\Phi}}{(1+2\Phi)}\right)\left(\frac{-2\dot{\Phi}}{(1+2\Phi)}\right)\right] \end{aligned} \quad (16)$$

The first two terms cancel. Simplifying the third term gives:

$$G_4^4 = \frac{3\dot{\Phi}^2}{(1+2\Phi)^3} \quad (17)$$

This should be zero as it is proportional to the energy density and there is no energy density inside the shell. This is only zero if Φ is *not a function of time*. Hence, Humphreys' assertion that the metric (9) is exact does not hold up. We note that this proof is in addition to the proof in equation (3) of section 2, which also showed that the metric inside the shell could not be *time dependent*.

Appendix B displays the entire Einstein tensor for both forms of Humphreys' interior metric. From those, we can detect the other unsupportable claim regarding the interior stress tensor, which will be discussed below.

The correct isotropic form of the spherically symmetric solution

In the previous section, we took a look at the Humphreys' claimed solution. It will be noted that his solution is in the *isotropic coordinate system*, where the form of the metric interval is:

$$ds^2 = -F(t,r)dt^2 + G(t,r)\left(dx^2+dy^2+dz^2\right) \quad (18)$$

However, the correct isotropic solution, as found in many texts,[11] is:

$$\begin{aligned} ds^2 &= -\left[(1-\varphi/2)/(1+\varphi/2)\right]^2 dt^2 + (1+\varphi/2)^4\left(dx^2+dy^2+dz^2\right) \\ \varphi(r) &= M/r \end{aligned} \quad (19)$$

This solution, in terms of a 'potential', is clearly not the solution claimed by Humphreys shown in eq. (9) above.

A major mathematical error

Humphreys comes to his conclusion partly due to a major mathematical error. The error occurs in equation (A44), where Humphreys claims "a simple transformation of the radial coordinate, … will eliminate *f(t)* from eq. (A43)". However, eq. (A44) is not a proper coordinate transformation.

Coordinate differentials (properly 1-forms on a manifold) must be exact. Humphreys writes, in eq. (A44):

$$d\bar{r}^2 = e^{f(t)} dr^2 \tag{20}$$

Or

$$d\bar{r}^2 = e^{f(t)/2} dr \tag{21}$$

showing that $d\bar{r}$ is not an exact differential. The correct way to perform coordinate transforms is to specify the new coordinates as functions of the old coordinates. This is an elemental principle of tensor calculus. Let y^a be the new coordinates and x^b the old coordinates. The correct transformation is to write:

$$y^a = \varphi^a(x) \tag{22}$$

The differentials then transform like contravariant vectors.

$$dy^a = \frac{\partial \varphi^a}{\partial x^b} dx^b \tag{23}$$

Since $\bar{r}$ is intended to be a function of t and r, it is correctly specified via:

$$\bar{r} = \varphi(t,r) \tag{24}$$

We then get the exact differential:

$$d\bar{r} = (\partial\varphi / \partial t)\, dt + (\partial\varphi / \partial r)\, dr \tag{25}$$

Not, $d\bar{r} = e^{f(t)/2} dr$ as in eq. (21).

This mathematical error and disregard for basic principles of tensor calculus is the precursor to Humphreys' final erroneous isotropic form of the metric in eq. (A59) and the conclusions drawn from it.

Misapplication of the Newtonian potential

We have shown, by way of several methods, that Humphreys' equations are not a solution of the EFE. As mentioned in section 1, Humphreys' claimed solution is based upon an illegitimate importing of a Newtonian potential into the metric tensor and an illegitimate imposition of a continuity condition. However, it is well known that potentials are only defined up to an arbitrary constant. The potential represents the work required to move an object against the force to which the potential is related. This is expressed in introductory physics texts as:

$$W = \int_A^B \vec{F} \bullet d\vec{r}$$

When the force is conservative, such as the gravitational field, the work is independent of path and the work can be used to define a potential that is a single valued function of position. Work is then measured by the *change* in potential.

$$\Delta U = U(\vec{x}_B) - U(\vec{x}_A) = -W = -\int_A^B \vec{F} \bullet d\vec{r} \tag{26}$$

This shows that the potential is defined relative to an *arbitrary* reference point (A) and an *arbitrary* value of the potential at that point. Due to the conservative nature of the force, the force is related to the potential via:

$$\vec{F} = -\nabla U$$

This also underscores the fact that adding an arbitrary constant to U has no physical consequences. The same gravitational field is represented. Choosing r = *infinity* and a value of zero for the potential there is arbitrary and merely a convention. *Changing the value of the reference potential by convention cannot cause changes in physical time dilation*.[12]

Instead of using infinity as the reference point and reference potential, one could choose the origin as the reference point and the reference potential. Using a reference potential of zero at the origin, equation (26) gives:

$$\frac{U}{m} \equiv \Phi(r) = \begin{cases} 0 & \text{if } r < r_0 \\ GM\left(\dfrac{1}{r_0} - \dfrac{1}{r}\right) & \text{if } r \geq r_0 \end{cases}$$

This potential yields the same *Newtonian physics* as choosing infinity as the reference. However, if one were to use this potential in the metric tensor one would obtain an entirely different physical 'effect'. In this case, clocks in the interior tick normally, while clocks at infinity tick more rapidly as the shell radius decreases. Neither of these two cases is true—they were both obtained by a faulty analogy. We also note that based on Humphreys' claims clocks can be made to tick slower or faster by way of an *action-at-a-distance* potential, i.e. superluminal. Rather than an improper mixing of non-relativistic Newtonian gravity with relativistic concepts, the correct result is obtained from properly using the GR formalism, resulting in equation (8) above.

All of the above are refutations of the main foundational claims of Humphreys' papers.[1,2] If they are invalid, the rest of the essay needs no more refutation. However, we now address a continuing theme in Humphreys' misunderstanding of the mathematical foundations of general relativity.

Which way to the future?

Humphreys continues to make the error regarding which coordinates represent time. This is apparent when he talks about 'imaginary times' and 'timeless zones' (or 'achronicity').

In the following, the space-like signature convention is used: (–, +, +, +). Angle brackets denote the invariant metric of the space-time. Consequently, a vector u pointing in the time direction would satisfy the invariant equation:

$$< u,u > < 0$$

Now let us consider a world line in Schwarzschild spacetime that is moving in the 't' direction of the Schwarzschild coordinates. Such a world line, in a general coordinate chart (not the Schwarzschild coordinates) labeled y^a, is given in terms of parametric equation:

$$y^a = y^a(t)$$

The vector tangent to this curve is:

$$u = u^a \frac{\partial}{\partial y^a} = \frac{\partial y^a}{\partial t}\frac{\partial}{\partial y^a}$$

Evaluation of the invariant norm gives:

$$\begin{aligned} <u,u> &= \left\langle u^a \frac{\partial}{\partial y^a}, u^b \frac{\partial}{\partial y^b} \right\rangle \\ &= \left\langle \frac{\partial}{\partial y^a}, \frac{\partial}{\partial y^b} \right\rangle \frac{\partial y^a}{\partial t}\frac{\partial y^b}{\partial t} \\ &= g_{ab} \frac{\partial y^a}{\partial t}\frac{\partial y^b}{\partial t} \end{aligned}$$

The last step used the definition of the metric components as inner products of the basis vectors in y coordinates. The last expression is the invariant norm in the coordinates given by y; which is precisely the transformation equation for the g_{tt} component of the metric in Schwarzschild coordinates. Thus:

$$<u,u> = g_{tt}^{\text{Schwarzschild}} = -\left(1 - \frac{2M}{r}\right) \tag{27}$$

Equation (27) is all that is needed. If $r > 2M$, then we see that inner product is less than zero and hence u is time-like in the exterior region outside the event horizon. However, if $r < 2M$ *(i.e. events inside the event horizon), then the inner product is positive. Thus, inside the event horizon* u *is space-like, and moving in the direction labelled by* t *is not a motion in time.*

These considerations also highlight the fact that once the shell crosses the coordinate singularity at the horizon, the equation for a material shell of matter of spatial radius r is no longer specifiable using the delta function given in equation (4) above.

At this point we also note that Humphreys' claim of a 'timeless' zone based on the metric in equation (9), above, is not only incorrect because of equation (27), but also because the other three dimensions (dx,dy,dz) become *temporal* due to the change in sign for $r < 2M$. The end result is that Humphreys' model would be spatially one dimensional and temporally three dimensional in the interior. Clearly, that does not qualify as a 'timeless' zone. The misinterpretation is again based on what seems to be a myopic fixation on the metric coefficient of the t coordinate and a seeming lack of rigorous geometric analysis. Of course, we repeat that equation (9) is not a solution of the field equations and so these conclusions are also based on an erroneous geometry.

A cavity solution in comoving coordinates

As was shown in Dennis,[5] spherically symmetric inhomogeneous models can be constructed in comoving coordinates. A good reference for this analysis is the seminal paper by H. Bondi.[4]

For the case of the cavity, we take the density $\rho(t,r)$ at some epoch denoted by proper comoving 'cosmic' time $t=0$ to be given by:

$$\rho(0,r) = \begin{cases} 0 \text{ if } r \le r_0 \\ \rho_0 \text{ if } r > r_0 \end{cases} \tag{28}$$

Here, ρ_0 is constant. In comoving coordinates the metric is:

$$ds^2 = -dt^2 + g_{rr}(t,r)dr^2 + R^2(t,r)d\Omega^2 \tag{29}$$

The coefficient of dt^2 is –1 since all clocks are radially free-falling at constant comoving coordinate r and thus register 'cosmic' time $dt^2 = -ds^2$. Note that $R(t,r)$ is no longer a radial coordinate but a function of the comoving coordinate r and the proper time t. However, the area of a sphere at time t and radius r is still $4\pi R^2(t,r)$.

Using eqns. (4)–(6) of Dennis[5] with $\dot{R}=\partial R/\partial t$ and $R'=\partial R/\partial r$ (the Bondi[4] equations with changes of notation):

$$\frac{1}{2}\dot{R}^2 - \frac{M(r)}{R} = E(r) \tag{30}$$

$$g_{rr}(t,r) = \frac{(R')^2}{1+2E(r)} \tag{31}$$

$$4\pi\rho(t,r) = \frac{M'(r)}{R'R^2} \tag{32}$$

Setting $E(r) = 0$ (i.e. we are taking the particles to be 'free'), we obtain:

$$\frac{1}{2}\dot{R}^2 - \frac{M(r)}{R} = 0 \tag{33}$$

$$g_{rr}(t,r) = (R')^2 \tag{34}$$

$$4\pi\rho(t,r) = \frac{M'(r)}{R'R^2} \tag{35}$$

The solution of the first equation for R is then:

$$\begin{aligned} R(t,r) &= \left[r^{3/2} + \sqrt{\frac{9M(r)}{2}}t \right]^{2/3} \\ &= r\left[1 + \sqrt{\frac{9M(r)}{2r^3}}t\right]^{2/3} \end{aligned} \tag{36}$$

We can integrate eq. (35) to obtain:

$$\begin{aligned} M(r) &= 4\pi \int \rho(t,r) d\left(\frac{1}{3}R^3\right) \\ &= \frac{4\pi}{3}\int \rho(0,r) d\left(R^3\right) \end{aligned} \tag{37}$$

Since $\rho(0,r)$ is piecewise constant we can move it outside the integrand, integrate over each constant region, and obtain:

$$M(r) = \begin{cases} 0 & \text{if } r \le r_0 \\ \frac{4\pi}{3}\rho_0\left(R(0,r)^3 - R(0,r_0)^3\right) & \text{if } r > r_0 \end{cases} \quad (38)$$

We can choose the definition of the comoving coordinate r so that:

$$R(0,r) = r\,,$$

obtaining:

$$M(r) = \begin{cases} 0 & \text{if } r \le r_0 \\ \frac{4\pi}{3}\rho_0\left(r^3 - r_0^3\right) & \text{if } r > r_0 \end{cases} \quad (39)$$

Note that, due to the cavity, an initially homogeneous density will become inhomogeneous over time. However, due to the nature of comoving coordinates, the mass $M(r)$ inside of comoving radius r is independent of time. This can be seen from the fact that no matter crosses the lines $r = constant$. The metric for the expanding matter outside the cavity is:

$$ds^2 = -dt^2 + \left(R'(t,r)\right)^2 dr^2 + r^2\left[1+\sqrt{\frac{9M(r)}{2r^3}}\,t\right]^{4/3} d\Omega^2 \quad (40)$$

Now consider the solution for $r < r0$. There $M(r) = 0$, and Eq. (36) becomes:

$$R(t,r) = r \quad (41)$$

So, the metric inside the expanding cavity reduces to:

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2 \quad (42)$$

i.e. static Minkowski space. This, as expected, is the same result as Eq. (5) and Eq. (8).

This solution is Minkowski space inside the cavity. Again, there is no 'timeless' zone. Clocks tick normally in the interior.

Birkhoff's theorem, exceptional claims and exotic energy

In this section we will analyze another aspect of the derivation of the shell metric presented in Appendix A of Humphreys.[1]

The derivation relies on an exceptional claim regarding the stress-energy and also fails to demonstrate a consistency check on the mathematics. The latter results in a remarkable assertion that there is a loophole in Birkhoff's theorem.

The exceptional claim regarding the stress energy is that it is of the form:

$$T^{tt} = T^{rr} = T^{\theta\theta} = T^{\varphi\varphi} = 0$$
$$T^{rt} \neq 0$$

In this section, to avoid the possibility of scribal lapses, we use the notation and equations from Appendix A of Humphreys.[1] The needed equations for the following analysis are (equation numbers by Humphreys):

$$G_t^t = -e^{-L}\left(\frac{L'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} = -8\pi T_t^t \quad \text{(A21)}$$

$$G_r^r = -e^{-L}\left(-\frac{N'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} = -8\pi T_r^r \quad \text{(A22)}$$

$$G_t^r = -e^{-L}\frac{\dot{L}}{r} = -8\pi T_t^r \quad \text{(A24)}$$

$$G_r^t = -e^{-N}\frac{\dot{L}}{r} = -8\pi T_r^t \quad \text{(A25)}$$

Eq. (A21) is equivalent to the third equation in eq. (2), above.

Integrating it, we can express L entirely in terms of G_t^t as follows:

$$\begin{aligned} G_t^t &= -e^{-L}\left(\frac{L'}{r} - \frac{1}{r^2}\right) - \frac{1}{r^2} \\ &= -r^{-2}\left[e^{-L}\left(L'r - 1\right) + 1\right] \\ &= -r^{-2}\left[-\left(re^{-L}\right)' + 1\right] \end{aligned}$$

Thus:

$$-r^2 G_t^t = -\left(re^{-L}\right)' + 1$$
$$\left(re^{-L}\right)' = 1 + r^2 G_t^t$$

Finally, integrating over r yields:

$$re^{-L} = r + \int r^2 G_t^t dr + C(t)$$

Thus:

$$e^{-L} = 1 + \frac{1}{r}\int r^2 G_t^t dr + \frac{C(t)}{r}$$

This functional form is not equivalent to Humphreys' shell metric. In fact, for there to be any resemblance, we would need to set $C(t)$ equal to zero, since it requires a globally time-dependent metric. Additionally, it requires an infinite singularity at the centre of the cavity. The centre singularity would indicate the presence of a time-dependent point mass at the origin, which *ex hypothesis*, is non-existent. For that reason, we set $C(t) = 0$. In addition, since Humphreys' shell model is static (correctly so) for $r > R(t)$, $C(t)$ must be set to zero.

Now, to show the inconsistency, we take the partial derivative of eq. (A21-D) with respect to time giving:

$$
\begin{aligned}
-e^{-L}\dot{L} &= \frac{1}{r}\int r^2 G^t_{t,t}\,dr + \frac{\dot{C}(t)}{r} \\
-e^{-L}\frac{\dot{L}}{r} &= \frac{1}{r^2}\int r^2 G^t_{t,t}\,dr + \frac{\dot{C}(t)}{r^2}
\end{aligned}
\qquad \text{(A21–D)}
$$

From eq. (A24), the LHS of the last equation is G^r_t thus we get:

$$
G^r_t = \frac{1}{r^2}\int r^2 G^t_{t,t}\,dr + \frac{\dot{C}(t)}{r^2}
$$

By Humphreys' assumptions G^t_t is zero inside the cavity (no mass density), thus:

$$
G^r_t = \frac{\dot{C}(t)}{r^2} = -8\pi T^r_t \qquad \text{(A24–D)}
$$

At this point we note two issues. First, eq. (A21-D) is not equivalent to Humphreys' Newtonian potential metric. Second, since C is independent of location, eq. (A24-D) represents a continuous flow of momentum *everywhere* with an infinite flux at the centre of the cavity. That is not a physically realizable configuration. At any rate, as noted in section 1, the terms of Humphreys' shell metric require $C(t)=0$. Thus, there are no off diagonal terms in the stress tensor.

Referring to Appendix B we see there that $G_{tr} = 0$, so that $T_{rt} = 0$.

In summary, the consistent metric for the shell cosmology is the one presented in eq. (8).

Claim that continuity of space requires continuity of the metric components

Humphreys[1] makes the following claims:

> "All the metric coefficients $g_{\mu\nu}$ are subject to a boundary condition that is very important to my argument. They must be continuous from just outside the shell all the way through to just inside it. Otherwise, spacetime (hence clocks and rulers) would change abruptly from one point to the next. That would be not only contrary to ordinary experience, but also hard to imagine theoretically in the absence of some extraordinary physical cause for it."

> "So, if $\dot{L}$ were zero in the cavity, g_{rr} could not vary with time. That conclusion conflicts with our previous conclusion in eq. (A17). Something must be wrong with the reasoning behind at least one of the two conclusions. Eq. (A17) stems straightforwardly from the continuity of spacetime and seems unassailable."[13]

There are two misconceptions in these quotes. The first is that the local coordinate metric components must be continuous throughout a manifold. In the second quote Humphreys' 'unassailable' argument conflates continuity of a local coordinate representation of a metric with 'continuity of spacetime' (however, note that the proper concept would be the *connectedness* of the spacetime). The former does not follow from the second. Below, we give an example of a two-dimensional manifold that is connected (but not *smooth*) which has discontinuous metric components. The dangers of not properly analyzing the geometries of manifolds with surface layers (as in the shell model) were pointed out by W. Israel[6] many years ago.[14] In short, Eq. (A17) is incorrect.

The second misconception is Humphreys' overlooking the fact that the shell model does contain "some … physical cause for it". The model contains a surface layer of mass at $R(t)$. This constitutes a Dirac delta contribution to the stress-energy tensor. By the EFE this implies that the Einstein tensor also contains a Dirac delta singularity. The result is that there is a discontinuity across the surface layer. The surface layer accounts for the fact that clocks outside the cavity are in a gravitational field (the Riemann tensor is non-zero outside). There, time dilation, relative to clocks at infinity and inside the cavity, takes place—it is due to the mass of the shell. Inside the cavity the Riemann tensor is zero, all clocks at rest in the interior tick at the same rate—there is no central mass inside the shell to influence clocks and rulers. Thus, there is an abrupt change in the spacetime geometry across the shell.

To demonstrate the independence of metric component continuity and spatial continuity we construct a two-dimensional manifold as in figure 1.

The surface consists of a conical section and a cylindrical section, as shown. The apex angle of the cone is α. We use cylindrical coordinates for the whole manifold, namely z and the azimuth angle ϕ. This manifold is everywhere flat except for a conical singularity at $z = 0$ and along the boundary where the extrinsic curvature is infinite in the z-direction, due to the discontinuous jump in the surface normal.

The boundary between the two regions occurs at $z = z_0$.

The metric interval in the cylindrical region is:

$$
ds^2 = dz^2 + R^2 d\phi^2
$$

in which, $R = \tan(\alpha) z_0$.

In the conical region $r(z) = \tan(\alpha)z$. This gives $dr = \tan(\alpha)dz$. So, on the cone, we get the induced interval:

$$
\begin{aligned}
ds^2 &= dz^2 + dr(z)^2 + r^2(z)d\phi^2 \\
&= \left(1 + \tan^2(\alpha)\right)dz^2 + \tan^2(\alpha)z^2 d\phi^2
\end{aligned}
$$

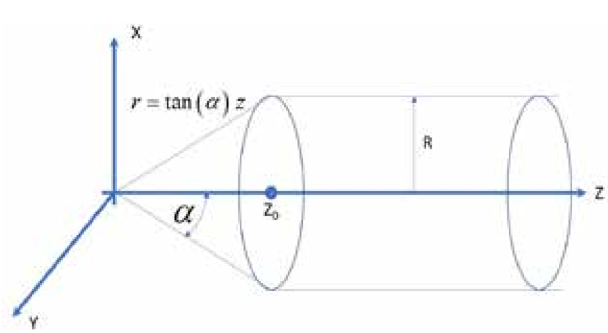

Figure 1. A surface of revolution with discontinuous metric

The complete specification of the metric interval is thus:

$$ds^2 = \begin{cases} \left(1+\tan^2(\alpha)\right)dz^2 + \tan^2(\alpha)z^2 d\phi^2 & \text{if } z \le z_0 \\ dz^2 + R^2 d\phi^2 & \text{if } z > z_0 \end{cases}$$

It is apparent that the manifold has continuity, yet the component g_{zz} is discontinuous across the boundary at $z = z_0$. As mentioned above there is an extrinsic curvature singularity at the boundary, which occurs at an abrupt change in the geometry.

Such surfaces can be easily multiplied by using the analysis of 'surfaces of revolution' covered in calculus.

Let the contour of the surface be specified as:

$$y = f(x)$$

We take f to be continuous but not necessarily the derivative $f' = \frac{df}{dx}$. The equation for the surface is:

$$ds^2 = \left(1+f'(x)^2\right)dx^2 + f(x)^2 d\phi^2$$

Immediately, we see that the metric coefficient g_{xx} is not continuous if the function f is only C^{1-}. We thus see that the 'unassailable assumption' of continuity of the metric components to ensure continuity of the manifold is false.

Conclusion and summary

We have seen the following points regarding Humphreys' shell-model:

1. The use of a Newtonian potential in GR is illegitimate. It is at odds with the theoretical foundations of GR.
2. The claim that metric components must be continuous is unfounded. We gave examples from GR and from elementary surfaces of revolution that refute Humphreys' postulated 'unassailable' principle.
3. The paper contains fundamental mathematical mistakes. The asserted solution can be shown to be incorrect by substitution into the EFE. Humphreys performs an erroneous 'coordinate transformation' to arrive at his putative 'isotropic solution'. The erroneous transformation contravenes the basics of tensor transformations that are central to GR.
4. There are numerous proofs that the metric inside a spherical cavity surrounded by an arbitrary external spherical distribution of matter is static flat space, i.e. Minkowski space. As a result, there are no timeless zones in the interior. These proofs were: (a) by direct integration of the EFE; (b) by substitution of Humphreys' metric into the EFE showing that the potential must be time independent inside the cavity; (c) by use of Birkhoff's theorem; and (d) by solving the EFE in comoving coordinates.

In summary, due to the above, I conclude that the shell-model is fatally flawed at the most fundamental levels of GR due to the paper containing mathematical errors and faulty conceptions.

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9. *The analysis above is sufficient to show that Newtonian potentials are inapplicable to the problem*. However, another point is that the mathematical structure of Newtonian gravity and the theory of general relativity are entirely different. Newtonian gravity is governed by a *time independent* equation for a *scalar* quantity. The gravitational field of GR is governed by *time dependent* equations for a second rank *tensor field*. Algebraically scalars and tensors are entirely different entities. They are not conceptually compatible. For example, the *spin* content (intrinsic angular momentum) of a scalar field is zero while the spin of a symmetric second rank tensor is two. Finally, the 'critical regions' (i.e. coordinate singularities) of a putative Newtonian potential are the limits at which *static* configurations cannot be maintained, thus also breaching the conditions for which Newtonian concepts are applicable. *In short, it is fallacious to employ non-relativistic physics to deduce relativistic conclusions. One needs to analyze within the confines of general relativity.*
10. It shouldn't be necessary to multiply counter examples to Humphreys' continuity claim, but the infinite discontinuity of $g_{rr} = \left(1-\frac{2M}{r}\right)^{-1}$ in the Schwarzschild metric at *r=2M* is another exception to the claim.
11. See MTW (1973), Exercise 31.7, p. 840.
12. To make this clear, the *exact general relativistic* formula for time dilation (for clocks at rest relative to the spatial coordinates) is $ds_1/ds_2 = \sqrt{g_{00}(1)/g_{00}(2)} = \sqrt{(1+2\Phi(1))/(1+2\Phi(2))}$. We have used Humphreys' substitution in the last equality. This formula shows that one cannot appeal to the Newtonian limit and add a constant Φ_0 to the potentials since $\sqrt{(1+2\Phi(1))/(1+2\Phi(2))} \neq \sqrt{(1+2\Phi(1)+2\Phi_0)/(1+2\Phi(2)+2\Phi_0)}$. Only in the weak field (Newtonian) limit is the last inequality approximately true, yet the argument cannot be substantiated.
13. Humphreys' equations (A16,17): $g_{tt} = g_{tt}(t), \quad g_{rr} = g_{rr}(t), \quad \text{for } r \le R(t)$
14. Israel[6] remarks: "In relativity, the ground is more slippery, because smoothness of the gravitational potentials $g_{\alpha\beta}$ is determined, not only by the smoothness of the physical conditions, but also by the smoothness with which the co-ordinates we happen to be using describe the space-time manifold. We are thus confronted with the not altogether trivial problem of disentangling the bumps arising out of the nature of the physical discontinuity from spurious bumps due to ill-matching of our coordinates at the surface in question."

Appendix A—Embedding equations for thin shell solution

We present the embedding equations for the thin-shell solution.

Two regions of space time: (1) Minkowski space in interior of thin spherical shell, denoted M_-; (2) Schwarzschild exterior to shell, denoted M_+. The exterior is the well-known Schwarzschild solution. The interior is Minkowski space.

In M_-: $ds^2 = -dT^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2$ A(43)

In M_+: $ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2(\theta) d\varphi^2$ A(44)

For ease of writing define $f(r) = \left(1-\frac{2M}{r}\right)$

The intrinsic metric of the shell hypersurface ∑ is:

$$ds^2 = -d\tau^2 + R^2(\tau) d\Omega^2. \quad A(45)$$

τ is the proper time along the geodesics. ∑ is a 3-dimensional time-like hypersurface, parameterized by the three coordinates:*(τ,Θ,φ)*. It represents the shell with time dependent radius in co-moving coordinates.

Embedding ∑ in the two regions is specified by functions that assign points in M_+ and M_- as a function of the hypersurface coordinates *(τ,Θ,φ)*:

$$M_+ : x_+{}^{\alpha} = X_+^{\alpha}(\tau,\theta,\varphi)$$

$$M_- : x_-{}^{\alpha} = X_-^{\alpha}(\tau,\theta,\varphi)$$

These are $x_-{}^{\alpha} = (T(\tau), R(\tau), \theta, \varphi)$ and $x_+{}^{\alpha} = (t(\tau), R(\tau), \theta, \varphi)$.

The components of the tangent vector to the time-like streamlines in the regions are:

$$M_+ : u_+^{\alpha} = \frac{dx_+{}^{\alpha}}{ds} = \left(\dot t(\tau), \dot R(\tau), 0, 0\right) \quad A(46)$$

$$M_- : u_-^{\alpha} = \frac{dx_-{}^{\alpha}}{ds} = \left(\dot T(\tau), \dot R(\tau), 0, 0\right) \quad A(47)$$

Normalization gives:

$$u_+^{\alpha} u_{+\alpha} = -1 = -\dot t^2(\tau) f(R) + \dot R^2(\tau) / f(R)$$

Therefore:

$$\dot t(\tau) \equiv \gamma_+(\tau) = \left[f(R) + \dot R^2(\tau)\right]^{1/2} / f(R)$$

Setting $M=0$ in this expression yields $\dot T(\tau)$

$$\dot T(\tau) \equiv \gamma_-(\tau) = \left[1 + \dot R^2(\tau)\right]^{1/2}$$

These expressions yield the transformations for the embeddings in M_+ and M_-:

$$t(\tau) = \int \gamma_+(\tau) d\tau = \int \frac{1}{f(R)} \left[f(R) + \dot R^2(\tau)\right]^{1/2} d\tau$$

$$T(\tau) = \int \gamma_-(\tau) d\tau = \int \left[1 + \dot R^2(\tau)\right]^{1/2} d\tau$$

$$r = R(\tau)$$

Using these in A(43) and A(44) yields the intrinsic interval in equation A(45), thus demonstrating consistency.

Appendix B—Einstein tensor for the Humphreys interior metric

Humphreys' interior metric in isotropic coordinates:

$$ds^2 = -(1+2\Phi(t))dt^2 + (1+2\Phi(t))^{-1}\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right) \quad B(1)$$

From this we compute the Einstein tensor (using Synge sign convention) for the interior.

$$G_t^t = \frac{3\left(\frac{\partial\Phi}{\partial t}\right)^2}{(1+2\Phi(t))^3} \quad B(2)$$

$$G_r^r = G_\theta^\theta = G_\varphi^\varphi = \frac{9\left(\frac{\partial\Phi}{\partial t}\right)^2 - 2(1+2\Phi(t))\frac{\partial^2\Phi}{\partial t^2}}{(1+2\Phi(t))^3} \quad B(3)$$

$$G_r^t = G_\theta^t = G_\varphi^t = G_\theta^r = G_\varphi^r = G_\varphi^\theta = 0 \quad B(4)$$

Since Humphreys claims $T_{tx} = T_{ty} = T_{tz} = 0$, we get $\Phi = 0$, contra Humphreys.

Also, $T_{tx} = T_{ty} = T_{tz} = 0$, contra Humphreys.

Humphreys' interior metric in curvature coordinates is:

$$ds^2 = -(1+2\Phi(t))dt^2 + (1+2\Phi(t))^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\varphi^2\right) \quad B(5)$$

Giving the Einstein tensor (Synge sign convention):

$$G_t^t = -\frac{2\Phi(t)}{r^2} \quad (B6)$$

$$G_r^t = -\frac{2\dot\Phi}{r(1+2\Phi(t))^2} \quad (B7)$$

$$G_r^r = -\frac{2\Phi(t)}{r^2} \quad (B8)$$

$$G_\theta^\theta = G_\varphi^\varphi = \frac{4\left(\frac{\partial\Phi}{\partial t}\right)^2 - (1+2\Phi(t))\frac{\partial^2\Phi}{\partial t^2}}{(1+2\Phi(t))^3} \quad (B9)$$

Since Humphreys claims $T_{tt} = T_{rr} = T_{\Theta\Theta} = T_{\varphi\varphi} = 0$, we get $\Phi = 0$, contra Humphreys.

This implies $T_{tr} = T_{rt} = 0$, contra Humphreys.

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