A biblical view restores reality to quantum mechanics

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The conventional view of quantum mechanics denies that particles have a definite position, momentum, and spin direction before those have been measured. A minority view (Appendix A) asserts that a particle always has definite values for those quantities, and that those 'hidden variables' influence the outcome of experiments measuring those quantities. Scripture appears to support the latter view. In the last few decades there have been many experiments trying to test which view is correct. They rely on a famous theorem by John Bell, called the Bell inequalities, to determine whether hidden variables exist or not. The conventional view violates the inequalities; Bell thought he proved that the minority view does not. Here I show that Bell's proof restricted the possibilities for hidden-variable theories and therefore does not apply to all such theories. I offer a counterexample, a hidden-variable model which gives the same correlation between widely separated detectors as orthodox quantum theory. This hidden-variable model violates Bell's inequalities, in just the same way as conventional quantum mechanics violates them. Recent experiments showing a violation of Bell's inequalities and conformity to standard quantum mechanics thus do not exclude the possibility of hidden variables. Therefore, the current alternative interpretation, 'entanglement' of particles, i.e. faster-than-light interaction between detectors, is not yet proven.

In 1928, Niels Bohr published[1] some of the ideas about quantum mechanics over which he and Einstein (figure 1) had their famous dispute at the 1927 Solvay conference in Brussels. Early in the paper Bohr wrote: "This [quantum] postulate implies a renunciation as regards the causal space-time coordination of atomic processes." Apparently, many theorists at the time took Bohr to mean that at least at the atomic scale, there can be no causal connection between two events with different spacetime coordinates. In particular, a particle could not have a definite trajectory between the points. He added: "Accordingly, an independent reality in the ordinary physical sense can neither be ascribed to the phenomena nor to the agencies of observation." In considering situations in which the specific orbit or trajectory of a particle is important, Bohr said: "we are bound to use the general solution of the wave equation which is obtained by superposition of characteristic solutions." Applying this to Heisenberg's newly published uncertainty principle, Bohr implied that, before measurement, a freely moving particle is generally in a mixture of quantum states and therefore would have no definite momentum or position. Of importance for later discussions, notice that this indefiniteness would also include the orientation of the particle's spin.

Biblical insight on the issue

We might expect the Bible to have something to say about whether the particles God created each have a definite existence (including position and velocity), or not. One verse is: "And there is no creature hidden from His sight, but all things are open and laid bare to the eyes of Him with whom we have to do" (Hebrews 4:13).

"All things" should include God's tiniest creations, the subatomic and subnuclear particles. For them to be seen would require them to have definite locations, I would think. Furthermore, Bohr's interpretation needs an observer to take note of a measurement before a particle can have a definite location or velocity. This verse says there is always an Observer who sees everything, even the smallest matter He created. Taken at face value, then, the Bible disagrees with Bohr's interpretation of reality. Similar thoughts may have prompted Einstein to say to Bohr at the 1927 conference: "God does not play dice [to determine the outcome of an experiment] with the universe!" Bohr reportedly replied: "Einstein, stop telling God what to do!" It would be interesting to hear what God might say on the matter, in addition to the verse above.

The controversy continues

In 1935, Einstein, Podolsky, and Rosen (EPR) took issue in print[2] with Bohr's ideas. They considered the case of two particles which are emitted simultaneously in opposite directions by a source, and then are measured with detectors far enough apart to disallow any interaction between them. Using standard quantum theory, EPR showed that a measurement of the momentum of one particle allows us to predict the momentum of the other particle. Or, if one detector determines the location of one particle, then we can predict the location of the other. According to EPR's objective definition

Figure 1. Niels Bohr (left) and Albert Einstein (right), December 1925

Image: Science Photo Library

Figure 2. John Stewart Bell, FRS

of 'reality,' this predictability means that both particles have real (definite) momenta and positions before they are detected. That contradicts Bohr's interpretation.

EPR did not comment directly on a related problem for Bohr, made clear by later authors: how could two detectors far enough apart not to interact nonetheless give correlated readings? J.S. Bell's comments[3] on some of Einstein's other writings about EPR indicate that Einstein thought the correlation between detectors would occur simply because each particle would maintain a definite trajectory (a definite momentum and position at every time) from source to detector, just as would be the case in classical mechanics.

In a letter to Einstein dated 7 June 1935, Erwin Schrödinger proposed the term 'entanglement' to describe the correlations between the two particles EPR described, and a few months later published a paper about it.[4] The term has come into wide use in recent decades. The idea today is that the wave functions of two particles can sometimes become tangled up with each other at the source. Then the joint wave function spreads out in all directions like a spherical cloud as the (non-located) particles move apart. When one particle materializes in a detector, the entangled wave function instantly transmits the information (faster than light) to the opposite detector and instructs the other particle to materialize there.

Shortly after that, Bohr replied[5] to the EPR paradox by suggesting that the very choice of apparatus at the detectors could somehow explain the correlation. Apparently, he was implying that if one chooses to measure momentum (or position) at one detector, one somehow gets a correlated momentum (or position) at the other detector. A half-century later, Bell found Bohr's reply hard to understand and suggested that Bohr may have been simply rejecting (without proof) EPR's premise of 'no action at a distance.'[6]

In 1952 David Bohm introduced an interpretation of quantum theory using what he termed 'hidden variables'.[7] He proposed that the quantum-mechanical wave function is a real field that would influence the motion of real particles, the particles having always a definite position, momentum, spin, and other characteristics. These quantities would be the hidden variables. In a companion paper,[8] he devoted a section to discussing the EPR *gedanken* (thought) experiment. Interestingly, Bohm did not appear to consider his hidden variables as explaining the correlations between the detectors. Instead he suggested the interaction occurs "instantaneously through the medium of the ψ-field [the wave function]", which sounds like the present-day idea of entanglement.

In 1964, John Bell (figure 2) published an article[9] about the EPR paradox that has become famous. He asserted that hidden-variable quantum theories can have certain qualities that could be checked experimentally. (He had just previously found that attempts to prove hidden variables impossible, such as a famous theorem by John von Neumann, were flawed.)[10] Many experiments have been done since then to test the existence of hidden variables. They all depend on the theorem Bell derived, in particular on several mathematical inequalities based on the theorem. The inequalities test the statistical correlations between two detectors located far from each other. Bell thought that all hidden-variable theories would obey the inequalities. In contrast, standard quantum theory would violate the inequalities. A recent experiment,[11] apparently without experimental loopholes, shows that the statistical correlation of its detector readings violates the Bell inequalities, and agrees with orthodox quantum theory. If all hidden-variable theories must indeed obey the inequalities, then the experiment would exclude hidden variables as an explanation for the correlations between detectors. The only other possibility so far suggested is that somehow each detector instantaneously interacts with the other one to produce the correlations, i.e. entanglement.

Bell's Theorem

Bell used an example, shown in figure 3, like one given by Bohm and Aharonov.[12] It depicts 'a pair of spin one-half particles [such as electrons or protons] formed somehow in the singlet spin state [spins opposite each other] and moving freely in opposite directions.' The particles can pass through Stern-Gerlach magnets (see Appendix B), beyond which detectors register either spin-up or spin-down counts (labelled +1 or -1 respectively) along the magnet axes **a** and **b**, each of which is a unit vector in various possible directions in a plane normal to the line from source to detector. Bell explained that if **a** is aligned with **b**, both quantum theory and experiment have the two particles having opposite spins in the detectors.

Next, Bell labelled the detector results A and B, and assumed that they are determined completely by the magnet axes **a** and **b** and a set λ of hidden variables:

$$A(\mathbf{a}, \lambda) = \pm 1 \ , B(\mathbf{b}, \lambda) = \pm 1 \qquad (1, 2)$$

Bell said that some of the set of hidden variables could be in common with both particles, but others of the set could be unique to each particle/detector. Note that A and B are step functions, not able to have any values between +1 and -1. Then Bell defined a correlation function $P(\mathbf{a}, \mathbf{b})$ (see Appendix C), which gives a number between +1 and -1 that tells how strongly the readings of detectors 1 and 2 should be related, depending on the relative orientations of **a** and **b**:

$$P(\mathbf{a},\mathbf{b}) = \int \rho(\lambda)\, A(\mathbf{a},\lambda)\, B(\mathbf{b},\lambda)\, d\lambda, \qquad (3)$$

saying '$\rho(\lambda)$ is the probability distribution of λ.' Bell compares eq. (3) to the correlation function that orthodox quantum mechanics would predict for the singlet state (opposite spins):[13]

$$P(\mathbf{a},\mathbf{b}) = \langle \boldsymbol{\sigma}_1 \cdot \mathbf{a} \ \ \boldsymbol{\sigma}_2 \cdot \mathbf{b} \rangle = -\, \mathbf{a} \cdot \mathbf{b} = -\cos\theta \qquad (4)$$

In this expression only, $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$ are the quantum-mechanical spin operators for particles 1 and 2, respectively, and θ is the angle between **a** and **b**. Bell then goes on to show that there is no way we can arrange for A and B in eq. (3) to depend on the set of hidden variables λ in order to get the right-hand side of eq. (4). The underlying reason for this impossibility turns out to be that A and B are step functions. Bell concluded that there should be a difference between the experimentally measurable correlation functions for hidden-variable theories and orthodox quantum mechanics. He thought all hidden-variable theories should obey his inequalities, whereas standard quantum theory violates the inequalities.

A hidden-variable model that is a counterexample

Bell thought his *ansatz* (initial assumption), equations (1, 2) with step functions, was a correct representation of all hidden-variable theories. I suggest that this does not cover all the possibilities for hidden variable theories. Below is a counterexample. It shows at least one hidden-variable model that gives a correlation function identical to that of orthodox quantum mechanics, $P(\mathbf{a},\mathbf{b}) = -\cos\theta$. So the model would violate the Bell inequalities, just as standard quantum theory does.

Imagine that each of the two particles in figure 3 has a definite location, momentum, trajectory, and unit spin vector $\boldsymbol{\sigma}$ as projected onto the plane of its detector. The source produces opposite spins in the two particles. Each particle is accompanied by waves that make it impossible to measure all these things precisely and simultaneously. The spin of each particle remains oriented in a particular direction all along its trajectory.

As figure 4 shows, the spin vector $\boldsymbol{\sigma}_1$ of particle 1 is at an angle of λ with respect to the Stern-Gerlach magnet axis (see Appendix B) for detector 1, vector **a**. The spin $\boldsymbol{\sigma}_2$ of particle 2 is at angle of $\lambda + \pi + \theta$ with respect to the magnet axis of detector 2, vector **b**. Each emission of a particle pair from the source produces a specific value of λ which remains the same all the way to the detector, but over many events λ can have any value between 0 and 2π. The correlation between detectors is produced by the source, with each particle retaining the spin information all along its path to the detector.

Here are the key differences I am introducing in this model. First, let us say that A and B are not step functions, but rather smoothly varying continuous functions whose *signs* give the reading (+1 or −1) of each detector:

$$\text{Reading of detector 1} = \text{sign}(A) \qquad (5)$$

$$\text{Reading of detector 2} = \text{sign}(B) \qquad (6)$$

(As Bell points out for a similar use of the sign function, the fact that the sign is undetermined for A or B being exactly zero makes no practical difference, since the probability of getting those exact values is zero). Using the sign function reflects the experimental observation that Stern-Gerlach magnets align the output spins with their axes, regardless of the initial spin directions of particles as they enter the magnets.

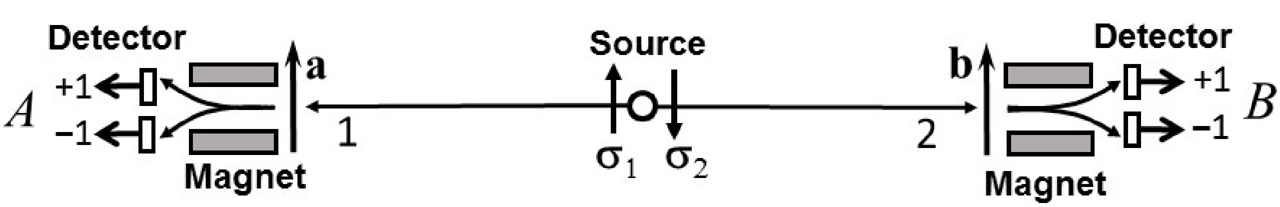

Figure 3. Bell's *gedanken* (thought) experiment has a source emitting two spin one-half particles (such as electrons or protons) in opposite directions. The spins, $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$, are opposite each other. The particles can pass through Stern-Gerlach magnets (see Appendix B) whose axes **a** and **b** can be oriented in any direction in planes perpendicular to the line between 1 and 2. The detector pairs remain lined up with the magnet axes. In Bell's analysis, A and B are step functions representing the outputs of the detectors, either +1 or −1.

That is, the output spins are either up or down with respect to the magnet axes. The amplitudes of A and B do not matter; they can be chosen for convenience. All the above replaces Bell's *ansatz*, equations (1, 2), with this one:

$$A(\mathbf{a}, \lambda, \alpha)\ ,\ B(\mathbf{b}, \lambda, \beta) \qquad (7, 8)$$

The main difference is that A and B are now continuous functions. A minor difference is that I have broken out from the set λ two subsets, α and β. These are the hidden variables that are unique to detectors 1 and 2, respectively. For example, α and β might specify just where and how a particle enters each detector. The symbol λ now represents the subset (in this case a single parameter, namely spin) of the hidden variables that both particles have in common. That is, λ travels with the particles. Turning now to the correlation function of eq. (3), Appendix C derives the function $\rho(\lambda)$ for this situation. It is:

$$\rho(\lambda) = \tfrac{1}{\pi} \qquad (9)$$

This value for $\rho(\lambda)$ turns out to normalize the correlation function, which we can now write in this form:

$$P(\mathbf{a},\mathbf{b}) = \tfrac{1}{\pi}\int_0^{2\pi} d\lambda\ \overline{A(\mathbf{a},\lambda,\alpha)}\quad \overline{B(\mathbf{b},\lambda,\beta)} \qquad (10)$$

The bars represent averages over the variables α or β, respectively. For this example of a hidden-variable dependence, I choose A and B to be such that the averages are:

$$\overline{A(\mathbf{a},\lambda,\alpha)} = \boldsymbol{\sigma} \cdot \mathbf{a} \qquad (11)$$

$$\overline{B(\mathbf{b},\lambda,\beta)} = -\,\boldsymbol{\sigma} \cdot \mathbf{b} \qquad (12)$$

This dot-product dependence is similar to that for fields in a polarized light beam transmitted through an analyzing filter. Figure 4 shows that the dot products in eqs. (11) and (12) depend on λ as follows:

$$\boldsymbol{\sigma} \cdot \mathbf{a} = \cos \lambda \qquad (13)$$

$$\boldsymbol{\sigma} \cdot \mathbf{b} = \cos (\lambda + \theta) \qquad (14)$$

Using eqs. (11) through (14) in eq. (10) gives:

$$P(\mathbf{a},\mathbf{b}) = -\tfrac{1}{\pi}\int_0^{2\pi} \cos \lambda\ \cos (\lambda + \theta)\, d\lambda \qquad (15)$$

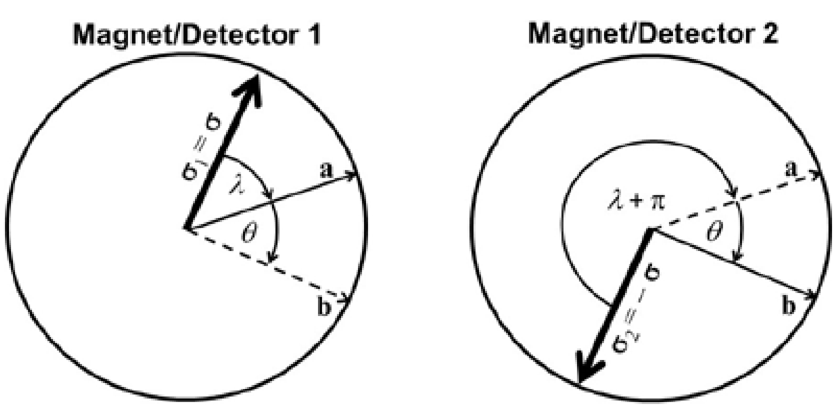

Figure 4. Orientations of spins ($\boldsymbol{\sigma}_1$, $\boldsymbol{\sigma}_2$), and magnet axes (**a**, **b**)

Evaluating the integral[14] gives the correlation function for this hidden variable model:

$$P(\mathbf{a},\mathbf{b}) = -\ \cos \theta \qquad (16)$$

This is the same as the result from orthodox quantum theory, eq. (4). That means this hidden-variable model would violate the Bell inequalities in just the same way as orthodox quantum mechanics violates the inequalities.

Conclusion

My counterexample shows that Bell's proof applies to only a subset of all possible hidden-variable theories. This means that experiments showing the correlation function has the orthodox form ($-\cos \theta$) do not disprove the possibility of hidden variables. Thus we should not yet conclude that entanglement and instantaneous interaction are realities. It could be that correlations between detectors are merely due to information imparted at the source and preserved in transit to the detectors. In retrospect, that seems to be a much less extraordinary explanation than faster-than-light interaction between particles alleged to be entangled. After all, it is merely Bohr's *interpretation* (called the Copenhagen interpretation) of quantum data which suggested that an undetected particle has no specific trajectory and therefore can carry no information in flight. Einstein, of course, would be glad to hear that Bohr has not yet been validated by experiment. And if it is true that Bell originally hoped hidden variables would prove to be reality, he might have been glad to hear that his theorem has a loophole.

Appendix A: The de Broglie–Bohm causal interpretation of quantum mechanics

Einstein never presented an explicit formulation of his idea that a particle should always have a definite position, momentum, and trajectory. One of the founders of quantum mechanics, Louis de Broglie, presented a paper with such a formulation at the 1927 Solvay conference. He called it the 'pilot-wave' theory, in which real waves would guide real particles. He later called it 'incomplete and diluted', and objections to it at the conference by Einstein and others caused him to set it aside. But he later returned to the theory, added much to it, and in 1960 published a book about it.[15] A little before that, David Bohm had published his work with similar ideas.[16] For the next three decades, a minority of physicists extended these ideas and collected them into a unified theory.[17]

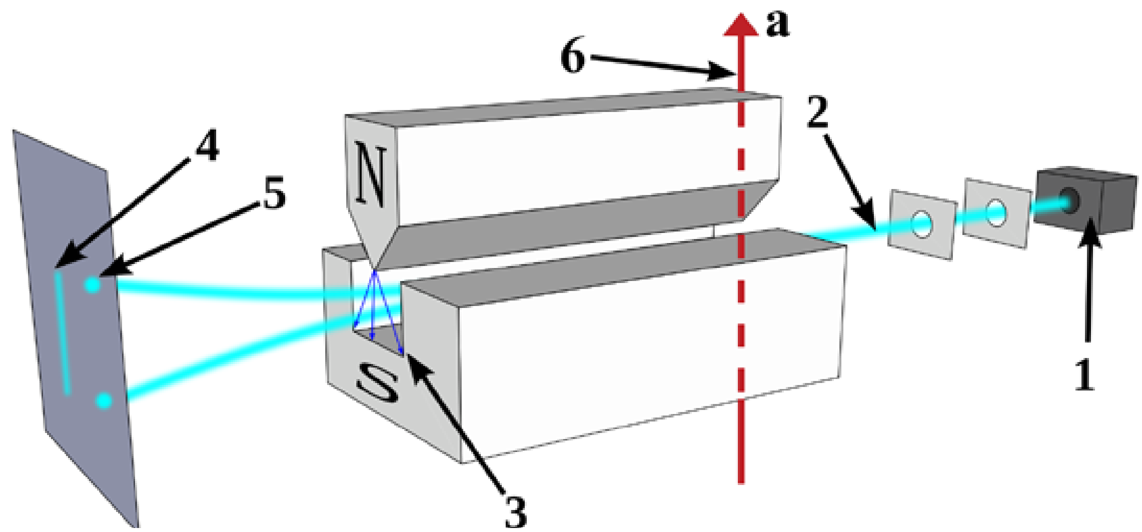

Figure 5. Stern–Gerlach experiment. Silver atoms travelling through a non-uniform magnetic field, and being deflected up or down depending on their spin: (1) furnace; (2) beam of silver atoms; (3) non-uniform magnetic field; (4) classically expected result; (5) observed result; (6) magnet axis, **a** (or **b**).

The basic idea is that particles always have a definite location and trajectory, and that they are accompanied by waves which are real, not mathematical constructs. (I add to the theory that the particles are the source of the waves.) The waves are able to influence the motion of the particles in a way that the theory specifies exactly. For example, in the famous 'two-slit' experiment, the particle goes through only one of the slits, but the waves go through both slits. Between the slits and the screen, the waves form an interference pattern. Depending on exactly where it goes through a slit, the particle will follow a path determined by the interference pattern. When it hits the screen, it contributes to an interference pattern on the screen that gradually becomes clear as more and more particles go through the slits. Theorists have plotted the sheaves of particle trajectories that result from the interference.[18,19]

Appendix B: The Stern–Gerlach experiment

Figure 5 shows the essentials of a well-known experiment done in 1922 by Walther Stern and Otto Gerlach.[20]

A silver atom has one unpaired electron in its outer electron shell, so it has the spin and magnetism (like a tiny bar magnet) of a single electron. Before they enter the gap between the big bar magnets, the spins of the silver atoms are oriented randomly. When an atom enters the gap, the quantum-mechanical waves accompanying the atom force the main part of its spin to be aligned either up (with the field) or down (against the field). The non-uniformity of the field then pulls the spin-up atoms upward and pushes the spin-down atoms downward. So the silver beam splits into two parts as in item 5 of the figure. Without the quantum-mechanical waves, the spins of the atoms in the gap would remain oriented randomly, and the beam would be a continuous sheet between up and down, as in item 4, the classically expected result. The magnet axis **a** (or **b**) in figure 3, figure 4, and the main text, is shown in item 6 of figure 5.

Appendix C: The correlation function and *ρ*(*λ*) for this case

From Bell's writings, it was difficult for me to decide what exact form for the correlation function my model should use, in particular what numerical value I should use for his function $\rho(\lambda)$ in eqs. (3) and (9). For example, he says: 'Let the correlation function be defined as the mean value of the product AB.'[21] Is there not an already agreed-upon definition? Then in the following equation he does not show $\rho(\lambda)$ at all. Is it 1, or is it subsumed in the averaging operation? So I decided to go for help to the branch of mathematics that developed the correlation function. Statistics textbooks define the correlation (in earlier texts called the 'correlation coefficient' and in this paper labelled P) between two variables x and y as:[22]

$$P(x,y) = \frac{1}{n\text{-}1} \sum \left(\frac{x_i - \bar{x}}{s_x}\right)\left(\frac{y_i - \bar{y}}{s_y}\right) \qquad \text{(C1)}$$

All the summations Σ here are for $i = 1$ to n. The values x_i and y_i are the i^{th} measurements of x and y in a set of n such measurements, $\bar{x}$ and $\bar{y}$ are the means of x_i and y_i, and s_x and s_y are the standard deviations of x_i and y_i. For x, the standard deviation is:[23]

$$s_x = \sqrt{\frac{1}{n\text{-}1} \sum (x_i - \bar{x})^2}\,, \qquad \text{(C2)}$$

and similarly for y. For my case below, it turns out that the means $\bar{x}$ and $\bar{y}$ are zero. Using those values and eq. (C2) applied to both x and y in eq. (C1) gives:

$$P(x,y) = \frac{\sum x_i y_i}{\sqrt{\sum x_i^2 \sum y_i^2}}\,, \qquad \text{(C3)}$$

The factors $1/(n{-}1)$ all cancel out. Now use eqs. (11) – (14) to substitute the variables of my model for x and y:

$$x = \overline{\mathrm{A}\,(\mathbf{a}, \lambda, \alpha)} = \cos\lambda\,, \qquad \text{(C4)}$$

$$y = \overline{\mathrm{B}\,(\mathbf{b}, \lambda, \beta)} = -\cos(\lambda+\theta) \qquad \text{(C5)}$$

Now let us specify n values of λ in order from 0 to 2π and call them λ_i. They are spaced a small angle $\Delta\lambda$ from each other, and we have $\Delta\lambda = 2\pi/n$. That gives us n values of x and y:

$$x_i = \cos\lambda_i, \quad y_i = -\cos(\lambda_i + \theta) \qquad \text{(C6, C7)}$$

Notice that the means of x_i and y_i are zero, as I said below eq. (C2). Put eqs. (C6) and (C7) into eq. (C3), and multiply all the sums by $\Delta\lambda$ (keeping the ratio the same), to get:

$$P(x,y) = \frac{-\sum \cos\lambda_i \cos(\lambda_i+\theta)\,\Delta\lambda}{\sqrt{\sum \cos^2\lambda_i\,\Delta\lambda \sum \cos^2(\lambda_i+\theta)\,\Delta\lambda}}\,, \qquad \text{(C8)}$$

Now take the limit of each sum as $\Delta\lambda \to 0$ *and* $n \to \infty$. That changes each sum to a definite integral:

$$P(\mathbf{a},\mathbf{b}) = \frac{-\int_0^{2\pi} \cos\lambda \cos(\lambda+\theta)\,\mathrm{d}\lambda}{\sqrt{\int_0^{2\pi} \cos^2\lambda\,\mathrm{d}\lambda \int_0^{2\pi} \cos^2(\lambda+\theta)\,\mathrm{d}\lambda}} \qquad \text{(C9)}$$

I have replaced x and y with **a** and **b** because λ, α, and β are integrated out of the result. Evaluating the two integrals in the denominator yields π for each one. That makes eq. (C9) become:

$$P(\mathbf{a},\mathbf{b}) = -\tfrac{1}{\pi}\int_0^{2\pi} \cos\lambda\ \cos(\lambda+\theta)\,d\lambda \qquad \text{(C10)}$$

This is identical to eq(15) in the main text, thus confirming eq. (9):

$$\rho(\lambda) = \tfrac{1}{\pi} \qquad \text{(C11)}$$

This result contradicts Bell's assumption that $\rho(\lambda)$ should be 'the probability distribution of λ,' for in that case ρ would be $1/2\pi$. Instead, the statistical definition of the correlation, eq. (C1), shows that ρ is:

$$\rho = 1/(s_A s_B)\,, \qquad \text{(C12)}$$

where s_A and s_B are the standard deviations of the distributions A and B. For Bell's choice of step functions for A and B, this would still give $\rho = 1/2\pi$. But for my choice of A and B in eqs. (11) and (12), ρ is $1/\pi$.

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